

Some title containing the words “homotopy” and
“symplectic”, e.g. this one¹

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There is an infinite series of notions, starting with symplectic manifolds ($n = 0$), Poisson manifolds ($n = 1$) and Courant algebroids ($n = 2$); there seems to be no name for higher n 's, so let us call the n 'th term Σ_n -manifolds. Their overview is in Table 1 at the end of the paper. Except for the non-standard terminology, this table is well known (the connection with variational problems may be an exception); nevertheless, it seems interesting to write a short informal review. We'll be mostly concerned with homotopy (or integration) of Σ_n -manifolds. The only non-trivial column is the one about quantization; for this reason, it won't be mentioned anymore.

The paper is based on a straightforward use of the basic idea of Sullivan's Rational homotopy theory [1] in differential geometry. Its connection with symplectic geometry is from [2].

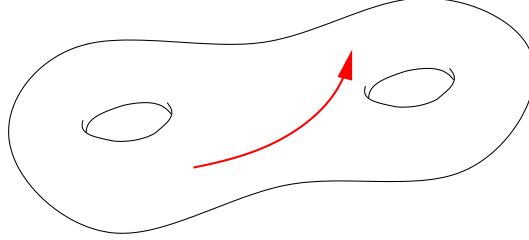
1 Integration of Lie algebroids (after Dennis Sullivan)

Let us begin with a simple construction of a groupoid Γ out of a Lie algebroid $A \rightarrow M$. Intuitively, A consists of infinitesimal morphisms of Γ ; to get all the morphisms, we have to compose them along curves. Thus, consider a

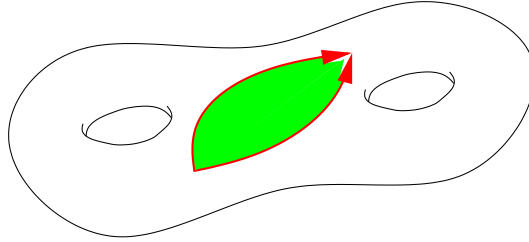
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Lie algebroid morphism $TI \rightarrow A$ (I is an interval), covering a curve $I \rightarrow M$.



It gives us a morphism in Γ between the endpoints of the curve. Two such morphisms $TI \rightarrow A$, with the same endpoints in M , give the same morphism in Γ , if they can be connected by a morphism $TD \rightarrow A$ (D is a disk, or better, to avoid problems with smoothness, a square with two opposite sides shrunk to points):



The composition in Γ is just concatenation (again, there is a problem with smoothness at the joint of the two paths—see below for a proper definition).

Let us notice how similar this construction of Γ is to the definition of fundamental groupoid. In what follows, we'll be looking at generalizations of Lie algebroids with non-trivial higher homotopies, and at their connections with symplectic geometry.

Finally, a little warning: the Γ we have defined need not be a manifold. The problem is that very large morphisms $TD \rightarrow A$ may connect very close morphisms $TI \rightarrow A$; thus, the equivalence classes may not be closed. On the other hand, when we only consider small morphisms, this construction gives a nice local Lie groupoid. (In the case when Γ happens to be a manifold, it is a Lie groupoid with 1-connected fibres.) Here is the moral: *beware of large disks*, or, more generally, *beware of large homotopies*.

For completeness, let us give a definition of Γ free of the smoothness problems. Two Lie algebroid morphisms $f_{1,2} : TI \rightarrow A$ will be called *homotopic rel boundary (r.b.)* if there is a morphism $g : T\Box \rightarrow A$ (where \Box is a square) such that the restrictions of g to $TJ_{1,2}$, where $J_{1,2}$ are the horizontal sides of \Box , are constant maps, while the restrictions to $TI_{1,2}$, where $I_{1,2}$ are

the vertical sides, are the morphisms $f_{1,2}$. Γ is the space of homotopy classes r.b. of morphisms $TI \rightarrow A$. A morphism $h_3 : TI \rightarrow A$ will be a composition of two morphisms $h_{1,2} : TI \rightarrow A$ if there is a morphism $k : T\Delta \rightarrow A$ (where Δ is a triangle) s.t. the restriction of k to the sides of Δ gives the three h 's (with the obvious orientation of the sides). Any two compositions are homotopic r.b., so the composition in Γ is well defined. Any two morphisms $TI \rightarrow A$ with a common endpoint in M can be composed, since there is a smooth retraction of Δ to a pair of its sides.

A closely related and equivalent construction (specialized to the case of Poisson manifolds) appeared in a paper of Cattaneo and Felder [4], who also considered the problem of smoothness of Γ by describing it as the leaf space of a foliation of an infinite-dimensional Banach manifold (of C^1 -morphisms $TI \rightarrow A$). Very recently the problem of smoothness of Γ has been completely solved by Crainic and Fernandes [5].

2 Some definitions from rational homotopy theory

Let us follow Sullivan further and make the following definitions: An *N-manifold* (shorthand for “non-negatively graded supermanifold”) is a supermanifold with action of the multiplicative semigroup (\mathbb{R}, \times) such that -1 acts as the parity operator (i.e. it just changes the sign of the odd coordinates). When we speak about the degree of a function (or of a vector field, etc.) on an N-manifold, we mean the weight of this action, i.e. $f(\lambda \cdot x) = \lambda^{\deg f} f(x)$, $\lambda \in \mathbb{R}$. Notice that the degrees of functions are always non-negative integers and that any function can be approximated (in the appropriate sense) by finite sums of homogeneous functions (since the degrees are just exponents of Taylor expansions around 0 in the semigroup \mathbb{R}). Finally, an *NQ-manifold* is an N-manifold with a degree-one vector field Q of square 0.

A basic example of an NQ-manifold is $T[1]M$, where M is an ordinary manifold (the functions on $T[1]M$ are the differential forms on M with their usual degrees, and Q is the de Rham differential). This will be our basic principle:

*Principle: we'll treat NQ-manifolds as if they were of the form $T[1]M$.*²

In particular, if N is a manifold and X an NQ-manifold, we'll treat NQ-maps $T[1]N \rightarrow X$ as if they were maps $N \rightarrow M$. Likewise, when we

²This principle is reasonable, as $T[1]$ is a full and faithful functor from manifolds to NQ-manifolds. It is also useful to notice that as a functor from N-manifolds to NQ-manifolds, $T[1]$ is the right adjoint of the corresponding forgetful functor.

to speak about their *homotopy*, we mean NQ-maps $T[1](N \times I) \rightarrow X$. If N is a manifold with boundary, an NQ-map $T[1](N \times I) \rightarrow X$ will be called a *homotopy rel boundary (r.b.)* if for any $x \in \partial N$ its restriction to $\{x\} \times T[1]I$ is a constant map.

A Lie algebroid structure on a vector bundle $A \rightarrow M$ is equivalent to an NQ-structure on the N-manifold $A[1]$. Notice that, according to our point of view, the groupoid Γ of §1 is really a fundamental groupoid.

NQ-manifolds can have interesting higher homotopies. To make a simple estimate, let us define the *degree* of an N-manifold X (a bit imprecisely, it is the highest degree of a coordinate of X). Let $x \in 0 \cdot X$ (notice that $0 \cdot X$ is an ordinary manifold, as -1 acts trivially there); x does not move under the action of (\mathbb{R}, \times) , so that the semigroup acts on $T_x X$; the highest weight of this action is the *degree at x* . It is locally constant in $0 \cdot X$. If $\deg X = 0$ then X is just an ordinary manifold (with trivial (\mathbb{R}, \times) -action). If X is of degree 1 then it is of the form $A[1]$ for some vector bundle A ; hence, an NQ-manifold of degree 1 is the same as a Lie algebroid.

Here is the estimate:

If X is an NQ-manifold of degree d , then X is “locally a homotopy d -type”, i.e. for any $n > d$, small NQ-maps $T[1]S^n \rightarrow X$ can be extended to small NQ-maps $T[1]B^{n+1} \rightarrow X$.

See Lemma 2 in the appendix for a more general (and more appropriate) formulation.

We can capture the local homotopy of an NQ-manifold using its fundamental n -groupoid, for $n \geq \deg X$: objects are points, 1-morphisms paths connecting the points, 2-morphisms disks connecting the paths, etc., and finally n -morphisms are homotopy classes of n -dimensional balls connecting $n-1$ -morphisms (recall that according to our principle, paths are NQ-maps $T[1]I \rightarrow X$ etc.; thus, e.g., points are just points in the manifold $0 \cdot X$). This n -groupoid is a reasonable generalization of the groupoid corresponding to a Lie algebroid (for a precise definition of n -groupoids via multisimplicial sets, and of fundamental n -groupoids in particular, see [3]). In what follows, we'll be also interested in various other objects that one can build using higher homotopies of NQ-manifolds (an example with nice pictures is the integration of a Lie bialgebroid to a double symplectic groupoid).

3 Examples: gerbes, loop groups, group-valued moment maps, etc.

In this section we'll be interested in NQ-manifolds very close to $T[1]M$, namely in principal $\mathbb{R}[n]$ bundles over $T[1]M$, $n \geq 1$ ($\mathbb{R}[n]$ is \mathbb{R} or $\mathbb{R}^{0|1}$, according to the parity of n , with the (\mathbb{R}, \times) -action given by $x \mapsto \lambda^n x$, and with $Q = 0$). The degree of such an X is n and $0 \cdot X = M$.

Lemma: Principal $\mathbb{R}[n]$ -bundles over $T[1]M$ (in the category of NQ-manifolds) are classified by $H_{DR}^{n+1}(M)$.

Indeed, in the category of N-manifolds, principal $\mathbb{R}[n]$ -bundles are trivializable, so let us choose an N-trivialization $X = T[1]M \times \mathbb{R}[n]$. The vertical part of Q on X can be identified with an N-map $T[1]M \rightarrow \mathbb{R}[n+1]$, i.e. with an $n+1$ -form η on M . Clearly, $Q^2 = 0$ iff $d\eta = 0$. Finally, changing the trivialization is equivalent to choosing an N-map $T[1]M \rightarrow \mathbb{R}[n]$, i.e. an n -form α ; η changes to $\eta + d\alpha$.

Let us also notice that when we choose an N-trivialization of X , an NQ-map $T[1]N \rightarrow X$ (where N is a manifold) is the same as a map $f : N \rightarrow M$ and an n -form ω on N such that $d\omega = f^*\eta$. We can see easily the n -groupoid corresponding to X : objects are points in M , 1-morphisms paths in M , etc., up to $n-1$ -morphisms; n -morphisms are not just homotopy classes of balls in M rel boundary (as they would be for $X = T[1]M$), but instead they are points in a principal bundle over this space of homotopy classes, with the structure group $\mathbb{R}/\{\text{periods of } \eta\}$ (notice, however, that $\{\text{periods of } \eta\}$ can be dense in \mathbb{R} ; in that case the n -groupoid is rather degenerate (recall: beware of large homotopies)).

Given a principal $U(1)$ -bundle over M , its Atiyah Lie algebroid gives us a principal $\mathbb{R}[1]$ -bundle over $T[1]M$. Vice versa, if the periods of such a $\mathbb{R}[1]$ -bundle are integers, we can integrate it to a groupoid, whose (any) fibre is a principal $U(1)$ (or \mathbb{R} , if all the periods vanish) bundle over the universal cover of M . There is a similar relation between $U(1)$ -gerbes and $\mathbb{R}[2]$ -bundles, etc.

3.1 Loop groups

This was a bit abstract and/or trivial, so let us pass to a nice concrete (and well studied) example. Let G be a simple compact Lie group with a chosen invariant inner product. The NQ-group $T[1]G$ has a central extension $1 \rightarrow \mathbb{R}[2] \rightarrow \widehat{T[1]G} \rightarrow T[1]G \rightarrow 1$ that is easily described in terms of its Lie algebra. Namely, the Lie algebra of $T[1]G$ is $\mathfrak{g} \oplus \mathfrak{g}[1]$ with zero commutator

in $\mathfrak{g}[1]$; Q maps $\mathfrak{g}[1]$ identically to \mathfrak{g} , and \mathfrak{g} to 0. The Lie algebra of $\widetilde{T[1]G}$ is $\mathfrak{g} \oplus \mathfrak{g}[1] \oplus \mathbb{R}[2]$, where the commutator in $\mathfrak{g}[1]$ is the inner product; Q acts on the $\mathfrak{g} \oplus \mathfrak{g}[1]$ -part as before, and sends $\mathbb{R}[2]$ to 0. This defines $\widetilde{T[1]G}$ as an NQ-group.

Notice that if M is a manifold, the space of NQ-maps $T[1]M \rightarrow \widetilde{T[1]G}$ is a group, and so is the space of homotopy classes of such maps.

Claim: The group of homotopy classes r.b. of disks in $\widetilde{T[1]G}$ coincides with the standard central extension \widetilde{LG} of the loop group.

Here, as before, a disk in $\widetilde{T[1]G}$ means an NQ-map $T[1]\mathbb{D} \rightarrow \widetilde{T[1]G}$ (where \mathbb{D} is a disk). The claim can be proved by an explicit calculation as follows. Notice that $G \subset \widetilde{T[1]G}$ (since $G = 0 \cdot \widetilde{T[1]G}$). The $\mathbb{R}[2]$ -bundle $\widetilde{T[1]G} \rightarrow T[1]G$ admits a both-sides G -invariant trivialization in the N-category that can be obtained by exponentiating the inclusion $\mathfrak{g} \oplus \mathfrak{g}[1] \subset \mathfrak{g} \oplus \mathfrak{g}[1] \oplus \mathbb{R}[2]$. A closed 3-form η should emerge from this trivialization, and indeed, it is the 3-form coming from the inner product on G , $\eta(u, v, w) = \langle u, [v, w] \rangle$. Therefore, an NQ-map $T[1]M \rightarrow \widetilde{T[1]G}$ is the same as a map $f : M \rightarrow G$ and a 2-form ω s.t. $d\omega = f^*\eta$. One can also check by differentiation that the product of two NQ-maps is given by $f = f_1 f_2$, $\omega = \omega_1 + \omega_2 + \langle f_1^* \theta_l, f_2^* \theta_r \rangle$, where $\theta_{l(r)}$ is the left (right) Maurer–Cartan form. Now the claim becomes the standard definition of \widetilde{LG} .

3.2 Moment maps

The group $\widetilde{T[1]G}$ is also connected with group-valued moment maps. Namely, let M be a G -manifold, so that $T[1]G$ acts on $T[1]M$. $T[1]G$ also acts on $\widetilde{T[1]G}$ by conjugation. Let us consider a $T[1]G$ -equivariant map $T[1]M \rightarrow \widetilde{T[1]G}$. As before, it can be described as a G -equivariant map $f : M \rightarrow G$, a 2-form ω s.t. $d\omega = f^*\eta$ plus one more condition on ω coming from the $T[1]G$ -equivariance. This is precisely the definition of G -valued moment map of [6], except for non-degeneracy of ω . Likewise, if $T[1]M_{1,2} \rightarrow \widetilde{T[1]G}$ are two NQ maps then their product $T[1](M_1 \times M_2) \rightarrow \widetilde{T[1]G}$ corresponds to the fusion product from the theory of G -valued moment maps.

Now we can describe a simple picture of general moment maps, following [7]. Let F be a group with invariant inner product and $G_{1,2} \subset F$ two Lagrangian subgroups (=half-dimensional & isotropic); the case of equivariant moment maps corresponds to $G_1 = G_2$. Notice that $T[1]G_{1,2} \subset \widetilde{T[1]F}$, as the G 's are isotropic. Let M be a G_1 -space; consider a $T[1]G_1$ -equivariant

NQ-map $T[1]M \rightarrow \widetilde{T[1]F/T[1]G_2}$. Again, we could describe it using a trivialization of the $\mathbb{R}[2]$ -bundle $\widetilde{T[1]F/T[1]G_2} \rightarrow T[1](F/G_2)$; it would become a G_1 -equivariant map $f : M \rightarrow F/G_2$, a 2-form ω , plus some conditions on ω .

Such maps can reasonably be considered as moment maps of general type. G -valued moment maps are included in a simple way: $F = G \times G$, the inner product is the difference of the inner products on the factors, $G_1 = G_2$ is the diagonal; we have $G \subset F$ (the first factor) preserving the inner product, and thus also $\widetilde{T[1]G} \subset \widetilde{T[1]F}$. Now we just identify $\widetilde{T[1]F/T[1]G_2}$ with $\widetilde{T[1]G}$.

As another example, suppose $G' \subset F$ is another Lagrangian subgroup, transversal to both $G_{1,2}$. We have $\mathbb{R}[2] \times T[1]G' \subset \widetilde{T[1]F}$ (G' is isotropic, so the central extension becomes trivial there), and we can identify (at least locally) $\widetilde{T[1]F/T[1]G_2}$ with $\mathbb{R}[2] \times T[1]G'$. A $T[1]G_1$ -equivariant NQ-map $T[1]M \rightarrow \widetilde{T[1]F/T[1]G_2}$ thus becomes G_1 -equivariant map $M \rightarrow G'$ (the G_1 action on G' comes from the identification of F/G_2 with G'), a closed 2-form, and a condition expressing the equivariance; this is the Lu-Weinstein moment map. In particular, if $F = T^*G_1$, and $G' = \mathfrak{g}_1^*$, we have the usual moment map (equivariant if $G_1 = G_2$, or twisted by a cocycle if not).

3.3 Variational problems

A reasonably complete understanding of variational problems requires symplectic geometry (see §5), but we can make few simple observations even now.

Let M be a manifold and N an n -dim manifold, and let us consider a Lagrangian for maps $N \rightarrow M$ that assigns a density on N to any map $N \rightarrow M$; the action of f is then the integral of this density (for simplicity, we shall always suppose that N is oriented, and we won't distinguish between densities and n -forms). We can say that the Lagrangian lifts maps $N \rightarrow M$ to NQ-maps $T[1]N \rightarrow T[1]M \times \mathbb{R}[n]$. As usual, we shall suppose that the density at any $x \in N$ depends only on the k 'th jet of the map $N \rightarrow M$ at x , where k is some fixed number.

However, as it often happens, a Lagrangian is defined naturally only up to a closed n -form on M . In fact, sometimes it is defined only locally, with a Čech 1-cocycle of closed n -forms on M giving the differences on the overlapping patches. Therefore, we should more properly say that the Lagrangian lifts maps $f : N \rightarrow M$ to NQ-maps $\tilde{f} : T[1]N \rightarrow X$, where

$X \rightarrow T[1]M$ is a principal $\mathbb{R}[n]$ -bundle. The action of f is then the homotopy class of f r.b.

For example, in the case of WZW model on a group G , $X = \widetilde{T[1]G}$. The fact that $\widetilde{T[1]G}$ is a group, plays an important role again: The solutions of the WZW model are $g(z, \bar{z}) = g_1(z)g_2(\bar{z})$. The maps $g_{1,2} : C_{1,2} \rightarrow G$ (where C 's are (complex) curves) have unique lifts to NQ-maps $\tilde{g}_{1,2} : T[1]C_{1,2} \rightarrow \widetilde{T[1]G}$ (just because C 's are curves); as it turns out, $\tilde{g} = \tilde{g}_1\tilde{g}_2$.

Finally, let us consider symmetries of variational problems. According to our principle (§2), the analogue of a 1-parameter group action $\mathbb{R} \times M \rightarrow M$ is an action $T[1]\mathbb{R} \times X \rightarrow X$ that is an NQ map. Infinitesimally, such an action is given by two vector fields ι and u , of degrees -1 and 0 respectively, such that $u = [Q, \iota]$ (so that, after all, we only need to know ι) and $[\iota, \iota] = 0$. Let now, as above, X be a principal $\mathbb{R}[n]$ -bundle over $T[1]M$, and let us consider only actions of $T[1]I$ preserving the bundle structure (i.e. commuting with the $\mathbb{R}[n]$ -action). Choosing a (local) NQ trivialization $X = \mathbb{R}[n] \times T[1]M$, ι can be encoded as a pair (v, α) , where v is a vector field and α an $n-1$ -form, both on M . The condition $[\iota, \iota] = 0$ means that $v \lrcorner \alpha = 0$.

Indeed, pairs (v, α) (regardless of the condition $v \lrcorner \alpha = 0$) appear as symmetries of n -dim Lagrangians in the usual formulation. Namely, the pair is a symmetry of Λ , if $\mathcal{L}_v \Lambda + d\alpha = 0$. It leads to a conservation law: if $f : N \rightarrow M$ is extremal, $d(\beta_v + f^* \alpha) = 0$, where β_v is certain $n-1$ -form on N (the momentum density with respect to v). Notice that the conserved momentum $\beta_v + f^* \alpha$ depends on α , not just on $d\alpha$. One could expect the pairs (v, α) to form a Lie algebra, but the bracket one obtains (the way how (v_1, α_1) acts on (v_2, α_2) , namely $[(v_1, \alpha_1), (v_2, \alpha_2)] \equiv ([v_1, v_2], \mathcal{L}_{v_1} \alpha_2 - v_2 \lrcorner d\alpha_1)$) is not skew symmetric for $n > 1$, and it only gives a Leibniz algebra. This is a small puzzle, but we can understand it easily now: the bracket is just $[u_1, \iota_2] = [[Q, \iota_1], \iota_2]$; since $[[Q, \iota], \iota] = 1/2[Q, [\iota, \iota]]$, it may be non-skew if $\deg X \geq 2$, when $[\iota, \iota]$ can be non-zero. For a complete description of symmetries we should pass to the differential graded Lie algebra of $\mathbb{R}[n]$ -invariant vector fields on X .

4 Σ_n -manifolds and their homotopy

A Σ_n -manifold is an NQ-manifold with a Q -invariant symplectic form of degree n . A Λ -structure is a Lagrangian NQ-submanifold. This section is based on [2].

If X is a Σ_n -manifold then $\deg X \leq n$. Indeed, if $x \in 0 \cdot X$, any weight k of the (\mathbb{R}, \times) -action on $T_x X$ appears with the same multiplicity as $n - k$,

so that $k \leq n$.

A Σ_0 -manifold is just a symplectic manifold and its Λ -structures are Lagrangian submanifolds. A Σ_1 -manifold is necessarily of the form $T^*[1]M$ where $M = 0 \cdot X$; Q has unique homogeneous Hamiltonian of degree 2 (or of degree $n + 1$ for Σ_n -manifolds), which is therefore a bivector field on M . In this way, a Σ_1 -manifold is the same as a Poisson manifold. Λ -structures are just conormal bundles of coisotropic submanifolds. Similarly, a Σ_2 -manifold is the same as a Courant algebroid. If $Y \subset X$ is a Λ -structure and moreover $0 \cdot Y = 0 \cdot X$ then Y is the same as a Dirac structure of the Courant algebroid.³

Let now M be a compact oriented n -dim manifold, possibly with boundary, and X a Σ_n -manifold, with symplectic form ω . The superspace of all maps $T[1]M \rightarrow X$ is symplectic, with symplectic form denoted ω_M , defined as follows: If $\psi : T[1]M \rightarrow X$ is any map, and $u, v \in \Gamma(\psi^*TX)$ (i.e. they are infinitesimal deformations of ψ), we let

$$\omega_M(u, v) = \int_{T[1]M} \omega(u(y), v(y)) dy,$$

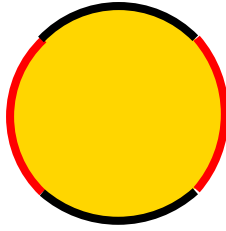
where dy is the natural Berezin volume form on $T[1]M$ (a function on $T[1]M$ is a differential form on M and the integral over $T[1]M$ is the integral of its top-degree part over M). Since we chose $\dim M = n$, we have $\deg \omega_M = 0$, so that the subspace of all NQ-maps is symplectic as well. If we restrict ω_M to the space of NQ-maps, it is no longer symplectic. However (according to Lemma 3 of Appendix), two NQ-maps lie on the same null leaf of ω_M iff they are homotopic r.b.; therefore, the space of homotopy classes r.b. is symplectic. Similarly, if we choose a Λ structure $Y \subset X$ and consider maps $T[1]M$ sending $T[1]\partial M$ to Y , the space of their homotopy classes is again symplectic. Finally, if $M = \partial M'$ for some M' , the homotopy classes of the NQ maps extendible to $T[1]M'$ form a Lagrangian submanifold (again Lemma 3), and in this way we get a symplectic analogue of $n + 1$ -dim topological field theory. One should interpret this with care, since it's based on formal computation of tangent spaces, cf. the beginning of the appendix.

Let us now pass to examples. The simplest Σ_2 -manifolds are of the form $\mathfrak{g}[1]$, where \mathfrak{g} is a Lie algebra with invariant inner product (any Σ_2 -manifold X s.t. $0 \cdot X$ is a single point is of this form). If M is a closed oriented surface, the NQ-maps $T[1]M \rightarrow \mathfrak{g}[1]$ are just flat \mathfrak{g} -connections on M , and

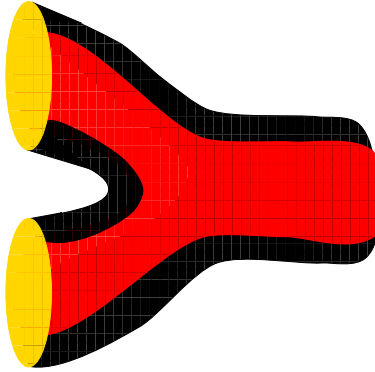
³Courant algebroids were defined by Z.J. Liu, A. Weinstein and P. Xu [8] to serve as Drinfeld doubles of Lie bialgebroids. Very different (and conceptually clear) doubles were defined A. Vaintrob (unpublished), using Σ_2 -manifolds (with double grading). The connection between the two approaches was clarified by D. Roytenberg [9] and A. Weinstein (unpublished).

the space of their homotopy classes is the moduli space of these connections, with its standard symplectic form. If M is a surface with boundary then the space of homotopy classes r.b. is the space of flat connections modulo gauge transformations vanishing at the boundary. In particular, if M is a disk, the space is \widetilde{LG}/G , which is a coadjoint orbit of \widetilde{LG} (in fact, $\mathfrak{g}[1]$ is a coadjoint orbit of $T[1]G$).

Let now $\mathfrak{h}_{1,2} \subset \mathfrak{g}$ be a Manin triple, so that $\mathfrak{h}_{1,2}[1]$ are mutually transversal Λ -structures in $\mathfrak{g}[1]$. There is a simple way of constructing the corresponding double symplectic groupoid (the symplectic analogue of the quantum group). The groupoid itself is the space of homotopy classes of maps from this disk to $\mathfrak{g}[1]$:



The black parts of the boundary are constrained to be mapped to $\mathfrak{h}_1[1]$ and the red parts to $\mathfrak{h}_2[1]$. (Again, speaking about maps from a disk, we mean NQ maps from $T[1]$ of the disk, etc.) The two multiplications are Lagrangian submanifolds of the third Cartesian power of the groupoid, and they are given by the following picture (this is one of them—to get the other, just exchange red and black):



The Lagrangian submanifold consists of the homotopy classes of those maps from three disks that can be extended to a map from the solid Y on the picture (with the boundary conditions as before). (Other operations, including the pictures of the Drinfeld double, can be found in [11].)

The same pictures can be used for integration of any Lie bialgebroid to a (local) double symplectic groupoid. We just need to recall the definition of Lie bialgebroid. It is a Σ_2 -manifold X with two Λ -structures $Y_{1,2}$ such that $0 \cdot Y_{1,2} = 0 \cdot X$ and $Y_1 \cap Y_2 = 0 \cdot X$. The procedure works as before, replacing $\mathfrak{g}[1]$ with X and $\mathfrak{h}[1]$ s with Y 's.

This simple construction has several variants. For example, we can turn one of the red arcs of the circle to pink, and consider one more Λ -structure $Z \subset X$ s.t. $0 \cdot Z = 0 \cdot X$; the new boundary condition is that pink is mapped to Z . One of the two multiplications survives recolouring, so that we still get a symplectic groupoid (though not double). According to Drinfeld [10], the quadruple (X, Y_1, Y_2, Z) gives rise to a Poisson homogeneous space; we have just constructed its symplectic groupoid.

We only dealt with Σ_2 -manifolds in our examples; many interesting things may be expected for higher n 's.

5 Σ_n -manifolds and variational problems (higher-dimensional Hamiltonian mechanics)

In this section we shall see that Σ_n -manifolds play much the same role in n -dim variational problems as Poisson manifolds do in classical mechanics. In classical mechanics, variational problems lead to Hamiltonian mechanics in (possibly twisted) cotangent bundles. More general Poisson manifolds appear most easily by reductions (or sometimes as degenerate limits). Nevertheless, Hamiltonian mechanics is interesting even if it doesn't arise in this way. We should mention one minor problem: 1-dim variational problems lead to Hamiltonian mechanics only under some invertibility condition (not satisfied e.g. for reparametrization-invariant Lagrangians). This can be circumvented by replacing the Hamiltonian—a function on a Poisson manifold P —with a Legendrian submanifold in the space of 1-jets of functions J^1P .

One can formulate n -dim generalization of Hamiltonian mechanics in the following way: Let Y be a Σ_n -manifold (recall that a Σ_1 -manifold is the same as a Poisson manifold). We'll be looking at NQ-maps $T[1]N \rightarrow Y$ (where N is an n -dim manifold) satisfying certain conditions (Hamilton equations). Λ -structures in Y will play the role of boundary condition, if we require (parts of) $T[1]\partial N$ to be mapped there.

Hamilton equations look as follows: let $x \in N$; let us choose a non-zero element of $\bigwedge^n T_x N$, so that we have a Berezin integral on $T_x[1]N$. Using this integral, the space of N-maps $T_x[1]N \rightarrow Y$ (denoted $NMap(T_x[1]N, Y)$)

is a symplectic manifold (finite-dimensional!). Its symplectic form depends linearly on the chosen element of $\bigwedge^n T_x N$, i.e. it is a symplectic form with values in the line $\bigwedge^n T_x^* N$. By definition, the *Hamiltonian'* at x , denoted H'_x , is a Lagrangian submanifold there. If $T[1]N \rightarrow Y$ is an NQ-map, *Hamilton equations* require its restriction to any $T_x[1]N$ to be in H'_x . In the case $n = 1$, $Y = T^*[1]P$ for some Poisson manifold P , and $NMap(\mathbb{R}[1], T^*[1]P)$ is just T^*P . As stated above, Hamiltonian is a Legendrian submanifold in J^1P ; H' is defined as its projection to T^*P , and we are back in the usual Hamiltonian mechanics.

For general n , the *Hamiltonian* at $x \in N$, denoted H_x , is a Legendrian submanifold in certain contact manifold C_x (a generalization of J^1P) which is a contactification of $NMap(T_x[1]N, Y)$. C_x is defined as follows: $NMap(T_x[1]N, Y)$ inherits the (\mathbb{R}, \times) action from Y , and therefore its symplectic form is exact in a canonical way (just plug the Euler vector field in); we just set $C_x = NMap(T_x[1]N, Y) \times \bigwedge^n T_x^* N$, with the contact structure given by the $\bigwedge^n T_x^* N$ -valued 1-form on $NMap(T_x[1]N, Y)$. Once H_x is chosen, H'_x is defined as its projection to $NMap(T_x[1]N, Y)$. Alternatively, one can define C_x by first contactifying Y to $Y^C = Y \times \mathbb{R}[n]$, setting $C_x = NMap(T_x[1]N, Y^C)$ and describing the way in which C_x inherits its contact structure from the contact structure on the NQ-manifold Y^C .⁴

Let us now describe how this picture arises from variational problems. We'll be interested in a Lagrangian that to any map $N \rightarrow M$ (where M is some manifold) associates a density on N ; moreover, we'll suppose that the Lagrangian is first order (i.e. the density depends on the first jet of the map $N \rightarrow M$ only). As we'll see, this situation is equivalent to the Hamiltonian mechanics described above, when we take $Y = T^*[n]T[1]M$; Hamiltonian and Lagrangian will become (after some identifications) the same object. As we discussed in §3.3, Lagrangian is natural only up to closed n -forms on M , and one should use a principal $\mathbb{R}[n]$ -bundle $X \rightarrow T[1]M$ to make it completely natural. In this picture we get $Y^C = j^1X$ (the space of first jets of sections of $X \rightarrow T[1]M$) and $Y = T^*[n]X//_1\mathbb{R}[n]$ (the symplectic reduction at moment 1). One can also go beyond Lagrangians for maps $N \rightarrow M$ and consider an $\mathbb{R}[n]$ -bundle $X \rightarrow Z$ over an arbitrary NQ-manifold Z of degree at most n (this is convenient in various kinds of gauge theories); one still

⁴If Z is a contact N-manifold of degree n and V an n -dim vector space then $NMap(V[1], Z)$ is a contact manifold: a vector $v \in T_f(NMap(V[1], Z))$ is in the contact hyperplane at f if v , when viewed as a section of f^*TZ , is actually a section of f^*CZ , where $CZ \subset TZ$ is the contact distribution on Z . Besides, contact NQ-manifolds are interesting objects, e.g. those of degree 1 are equivalent to manifolds with Jacobi structure.

gets $Y^C = j^1 X$ and $Y = T^*[n]X // {}_1\mathbb{R}[n]$ (if X is trivial then $Y = T^*[n]Z$).

This is how it happens: In the usual terms, a Lagrangian at $x \in N$ is a $\bigwedge^n T_x^* N$ -valued function on the space of linear maps $T_x N \rightarrow TM$, i.e. on the space $NMap(T_x[1]N, T[1]M)$. If we use a principal $\mathbb{R}[n]$ -bundle $X \rightarrow T[1]M$, a Lagrangian at x is a section of the principal $\bigwedge^n T_x^* N$ -bundle $NMap(T_x[1]N, X) \rightarrow NMap(T_x[1]N, T[1]M)$. Finally, to include multivalued and/or partially defined Lagrangians, we should define them as Legendrian submanifolds in the space of 1-jets of sections of this bundle, $j^1 NMap(T_x[1]N, X)$. We just notice the natural isomorphism $j^1 NMap(T_x[1]N, X) = NMap(T_x[1]N, j^1 X)$; in this way we get to Hamiltonian mechanics, with $Y^C = j^1 X$ and with Hamiltonian equal (using the isomorphism) to the Lagrangian.

One question concerning our higher-dimensional Hamiltonian mechanics certainly remains: *what is the meaning of all that?* We give a rather incomplete answer (incomplete = not quite true; please take it with a heap of salt); perhaps this can be excused by the importance of the question. We pass from N to any n -dim submanifold with boundary (and corners) $N_1 \subset N$; the space of homotopy classes r.b. of NQ-maps $T[1]N_1 \rightarrow Y$ is symplectic (as we saw in §5) and those obeying Hamilton equations form a Lagrangian submanifold. When we decompose N_1 as $N_2 \cup N_3$, these Lagrangian submanifolds compose in the natural way, so that it's sufficient to know them for infinitesimal bits of N . The Hamiltonian at $x \in N$ encodes such a Lagrangian submanifold for an infinitesimal bit around x . This picture could be quite interesting if it survives quantization.

6 Example: geometry of non-abelian conservation laws in two dimensions

A conservation law in an n -dim variational problem means that we are given an $n - 1$ -form which is closed if our configuration is extremal. For $n = 2$ there is a natural non-abelian generalization: instead of 1-forms, we can consider \mathfrak{g} -connections (for some Lie algebra \mathfrak{g}) that are flat on extremals. One can consider non-abelian generalizations for higher n 's as well: a flat connection is an NQ-map to $\mathfrak{g}[1]$, and in principle we can replace $\mathfrak{g}[1]$ with any NQ-manifold. However, the geometry of these general conservation laws is unclear to me; we shall restrict ourselves to $n = 2$ and to $\mathfrak{g}[1]$.

Thus, we shall consider maps from a surface N to a manifold M ; any such map should produce a \mathfrak{g} -connection on N , flat if the map is extremal. More naturally, we'll be given a principal G -bundle $P \rightarrow M$; any map $f : N \rightarrow M$

should produce a connection on f^*P , flat if f is extremal. We will call this a *P-conservation law*. A flat connection on f^*P is the same as a lift of f to an NQ-map $T[1]N \rightarrow (T[1]P)/G$.

The idea is as follows: The variational problem will be reformulated as 2dim Hamiltonian mechanics in a Σ_2 -manifold Y_M . We shall then define a coisotropic NQ-submanifold $Y_{M|0} \subset Y_M$ and an NQ-map $Y_{M|0} \rightarrow (T[1]P)/G$. If the Hamiltonian is such that the Hamilton equations force maps $T[1]N \rightarrow Y_M$ to be maps $T[1]N \rightarrow Y_{M|0}$ then we have a *P-conservation law*. Since $Y_{M|0}$ is coisotropic, this condition is easily understood. Recall that for any $x \in N$, H'_x is a Lagrangian submanifold in $NMap(T_x[1]N, Y_M)$; we want it to be a submanifold of $NMap(T_x[1]N, Y_{M|0})$. Since $NMap(T_x[1]N, Y_{M|0})$ is a coisotropic submanifold of $NMap(T_x[1]N, Y_M)$, it just means that H'_x is a maximally isotropic submanifold of the presymplectic manifold $NMap(T_x[1]N, Y_{M|0})$; in particular, it is woven of the null leaves of the presymplectic form. In the favorable case, when the space of null leaves is a supermanifold (hence a Σ_2 -manifold), it reduces to a Hamiltonian on this space of leaves.

$Y_{M|0} \subset Y_M$ and $Y_{M|0} \rightarrow (T[1]P)/G$ will be constructed via a group action, and the condition on Hamiltonian will be reformulated as its invariance, i.e. we get a non-Abelian version of Noether theorem. Namely, consider a Lie algebra \mathfrak{f} with invariant inner product, containing \mathfrak{g} as a Lagrangian subalgebra (i.e. $\mathfrak{g} \subset \mathfrak{f}$ is a Manin pair). We shall suppose that F acts on P , extending the action of G (the action of F needn't preserve the bundle structure). In addition, we shall also need a principal $\mathbb{R}[2]$ -bundle $X_P \rightarrow T[1]P$ (in NQ-category) and an NQ-action of $\widetilde{T[1]F}$ on X_P , covering the action of $T[1]F$ on $T[1]P$ and extending the action of $\mathbb{R}[2]$ on X_P . We let $X_M = X_P/T[1]G$ (thus $X_M \rightarrow T[1]M$ is a principal $\mathbb{R}[2]$ -bundle). As in §5, we define $Y_{P,M}$ as $T^*[2]X_{P,M}/_1\mathbb{R}[2]$. Clearly $Y_M = Y_P//T[1]G$.

The group $\widetilde{T[1]F}$ acts on Y_P ; as $\mathbb{R}[2] \subset \widetilde{T[1]F}$ acts trivially, the action factors through a (non-Hamiltonian) $\widetilde{T[1]F}$ -action. The moment map μ of the $\widetilde{T[1]F}$ -action takes values in $T^*[2](\widetilde{T[1]F})/\widetilde{T[1]F} = \mathbb{R} \oplus \mathfrak{f}^*[1] \oplus \mathfrak{f}^*[2]$; the \mathbb{R} -component of μ is identically 1. Let $\mu_0 = 1 \oplus 0 \oplus 0 \in \mathbb{R} \oplus \mathfrak{f}^*[1] \oplus \mathfrak{f}^*[2]$; μ_0 is clearly a fixed point of Q and it is also (\mathbb{R}, \times) -invariant, so $Y_{P|\mu_0} = \{x \in Y_P, \mu(x) = \mu_0\} \subset Y_P$ is a NQ-manifold and so is the symplectic reduction $Y_P//_{\mu_0}\widetilde{T[1]F} = Y_{P|\mu_0}/F$, if it is a supermanifold (F is the subgroup of $\widetilde{T[1]F}$ fixing μ_0). Notice that $Y_{P|\mu_0}/G \subset Y_M$ is a coisotropic NQ-submanifold (as $Y_M = Y_P//T[1]G$). We let $Y_{M|0} = Y_{P|\mu_0}/G$. If $Y_P//_{\mu_0}\widetilde{T[1]F}$ is a supermanifold, it is the space of null leaves of $Y_{M|0}$. The NQ-map $Y_{M|0} \rightarrow (T[1]P)/G$

needed for P -conservation law comes from the projection $Y_P \rightarrow T[1]P$.

To conclude, we should explain how our condition on the Hamiltonian can be expressed as its symmetry. We define an auxiliary symplectic N-submanifold $Y_{P|>}$ of Y_P as the subspace where the $\mathfrak{f}^*[1]$ -component of μ vanishes. The group F acts on $Y_{P|>}$ and its moment map vanishes on $Y_{P|0}$. Let H_x^P be a Legendrian submanifold in $NMap(T_x[1]N, Y_{P|>}^C)$; it can be projected to a Legendrian submanifold in $NMap(T_x[1]N, Y_M^C)$, i.e. to a Hamiltonian in Y_M , iff it is G -invariant, and the Hamiltonian will satisfy our condition iff H_x^P is F -invariant.

Finally, there is one more interesting phenomenon concerning P -conservation laws. Suppose there is another Lagrangian subgroup $\bar{G} \subset F$ such that $P \rightarrow P/\bar{G}$ is a principal \bar{G} -bundle. Then we can define another variational problem with target $\bar{M} = P/\bar{G}$, just by projecting H_x^P to a Legendrian submanifold of $NMap(T_x[1]N, Y_{\bar{M}}^C)$. Moreover, one can transfer solutions of Hamilton equations between Y_M and $Y_{\bar{M}}$: Recall that our Hamilton equation constrain NQ-maps $T[1]N \rightarrow Y_M$ to be NQ-maps $T[1]N \rightarrow Y_{P|\mu_0}/G$. The latter maps can be lifted to NQ-maps $T[1]N \rightarrow Y_{P|\mu_0}$ (provided $\pi_1(N) = 0$, otherwise the lift may be multivalued) and these lifts are unique up to G -action, i.e. the projection $Y_{P|\mu_0} \rightarrow Y_{P|\mu_0}/G$ is a “normal covering with group G ” (this is because G is “discrete” in the world of NQ-manifolds, i.e. for any manifold K , any NQ-map $T[1]K \rightarrow G$ is constant). Then we project this lift to an NQ-map $T[1]N \rightarrow Y_{\bar{M}|0}$. This procedure takes solutions to solutions and in this way the two variational problems become equivalent. This is the *Poisson-Lie T-duality* of [12].

Appendix: Infinitesimal deformations and vector bundles

Let X and Y be NQ-manifolds; recall that by a homotopy connecting two NQ-maps $Y \rightarrow X$ we mean an NQ-map $Y \times T[1]I \rightarrow X$. We’ll be interesting in the NQ-maps $Y \rightarrow X$ close to a given NQ-map $\psi : Y \rightarrow X$, and also in the question which of them are homotopic to each other by small homotopies. We will simply linearize the problem (considering infinitesimally close maps) and compute formally the tangent space to the space of homotopy classes of NQ-maps $Y \rightarrow X$. It is by no means true that this space is always a manifold and we don’t give any criteria for the formal tangent space to be actual tangent space. This is a serious flaw in this paper.

An infinitesimal deformation of $\psi : Y \rightarrow X$ is a section of ψ^*TX ; the infinitesimal part of a homotopy starting at ψ is a linear homotopy con-

necting a section of ψ^*TX with 0. By an *NQ vector bundle* $E \rightarrow Y$ we mean a vector bundle that is both (\mathbb{R}, \times) - and Q -equivariant (ψ^*TX was an example). Notice that the vector superspace of its sections is a cochain complex (a section s is of degree d if $s(\lambda \cdot x) = \lambda^d(\lambda \cdot s(x))$, $\lambda \in \mathbb{R}$, where \cdot denotes the (\mathbb{R}, \times) -action; d may be negative, but it's bounded from below, e.g. $d \geq -\deg X$ for $E = \psi^*TX$). A section is an N-map if it's of degree 0, it is an NQ-map if it's also closed, and it is connected with the zero section by a linear homotopy if it's exact. Notice however that *any* homotopy in the space of sections can be made to a linear homotopy. Hence, $H^0(\Gamma(E))$ is the space of homotopy classes of NQ sections, whether the homotopies are required to be linear or not.

Negative cohomologies have a homotopical meaning as well; namely, $H^{-n}(\Gamma(E))$ is “ π_n of the space of sections”. More precisely, since $H_{DR}(B^n, S^{n-1}) = \mathbb{R}[n]$, we have:

Lemma 1: Let $E \rightarrow Y \times T[1]B^n$ be an NQ vector bundle, and let E_0 be its restriction to Y (where Y is embedded into $Y \times T[1]B^n$ as (say) $Y \times \{\text{centre of the ball}\}$). Let $\Gamma_0(E)$ denote the sections of E vanishing at $Y \times T[1]S^{n-1}$. Then $H(\Gamma_0(E)) = H(\Gamma(E_0))[n]$.

This lemma can also be generalized to a form of Thom isomorphism (replacing $Y \times T[1]B^n$ by a bundle).

Now, let $\psi : T[1]B^n \rightarrow X$ be an NQ-map. Setting $Y = \{pt\}$, $E = \psi^*TX$, the previous lemma gives us a linearized version of the following:

Lemma 2: Let $\psi : T[1]B^n \rightarrow X$ be an NQ-map. If $n > \deg X$ then any NQ-map close to ψ and coinciding with ψ on $T[1]S^{n-1}$ is homotopic to ψ r.b. by a small homotopy, i.e. X is “locally a homotopy $\deg X$ -type”.

It is possible to deduce this claim from its linearized version (a nice explanation would be nice).

Let us now pass to symplectic vector bundles. A Σ_n *vector bundle* is an NQ vector bundle with symplectic fibres, with the field of symplectic forms of degree n , and with Q compatible with the symplectic structure.

We'll be mostly interested in Σ_n bundles $E \rightarrow T[1]M$, where M is an oriented compact manifold, possibly with boundary. Notice that $\Gamma(E)$ is a symplectic vector superspace; the symplectic form (denoted ω_M) is given by

$$\omega_M(u, v) = \int_{T[1]M} \omega(u(y), v(y)) dy,$$

where dy is the standard Berezin volume form on $T[1]M$. The degree of ω_M is $n - \dim M$.

Let E' be the restriction of E to $T[1]\partial M$ and u', v' the restrictions of u

and v . Notice (by Stokes theorem) that

$$\omega_{\partial M}(u', v') = \pm \omega_M(Qu, v) \pm \omega_M(u, Qv).$$

As a consequence,

Lemma 3: $Z\Gamma(E) = (B\Gamma_0(E))^\perp$ (where $\Gamma_0(E)$ denotes the subcomplex of sections vanishing at $T[1]\partial M$), so that $Z\Gamma(E)/B\Gamma_0(E)$ is symplectic. In particular, if M is closed, $H(\Gamma(E))$ is symplectic (a version of Poincaré duality). The image of $H(E) \rightarrow H(E')$ is a Lagrangian subspace.

Notice that if $\dim M = n$, $\deg \omega_M = 0$, so that $(Z\Gamma(E)/B\Gamma_0(E))^0$, the space of homotopy classes r.b. of NQ-sections of E , is symplectic. Finally notice that if M is not closed, $H(E)$ is not symplectic. But there is a simple way how to make Q selfadjoint. We choose a Lagrangian NQ subbundle $L \subset E'$, and let $\Gamma_L(E)$ be the sections taking values in L over $T[1]\partial M$. Then $H(\Gamma_L(E))$ is symplectic.

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n		<i>substructures</i>	<i>integration = homotopy</i>	<i>quantization</i>	<i>variational problems</i>
0	symplectic manifolds	Lagrangian submanifolds	—	vector spaces	—
1	Poisson manifolds	coisotropic submanifolds	symplectic groupoids	associative algebras, abelian categories	particle mechanics
2	Courant algebroids	(generalized) Dirac structures	symplectic 2-groupoids, double symplectic groupoids, 3-dim symplectic TFTs, etc.	abelian 2-categories, quantum groups, 3-dim TFTs	2-dim variational problems
n	Σ_n -manifolds	Λ -structures	symplectic n -groupoids, $n + 1$ -dim symplectic TFTs, etc.	abelian n -categories, $n + 1$ -dim TFTs	n -dim variational problems

Table 1
Whatever this table is about